Proceedings of the Royal Society of Edinburgh, **88A**, 83-92, 1981

Existence and multiplicity result for a class of second order elliptic equations

Helmut Hofer
Mathematics Institute, University of Zurich, Freiestrasse 36, CH-8032 Zurich, Switzerland

(MS received 23 January 1980. Revised MS received 30 June 1980)

Synopsis

Some second order semilinear elliptic boundary value problems of the Ambrosetti-Prodi-type are studied. Existence and multiplicity of solutions is proved in dependence on a parameter. Constructing a global strongly increasing fixed point operator in a suitable function space, observing – under appropriate conditions, which are in some sense optimal – that the fixed point operator has some properties similar to a strongly positive linear endomorphism, one unifies and improves the treatment of such problems, whether the nonlinearity is dependent on the gradient or not, and obtains some new results.

I. Introduction and statement of the results

In this paper we study the semilinear boundary value problem (BVP)

$$(\mathrm{P}_t) \qquad \begin{aligned} Lu &= G(x, u, Du) + tr \quad \text{in } \Omega \\ Bu &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where t is a real parameter value. Here Ω denotes a bounded domain in $\mathbb{R}^n$ whose boundary $\partial\Omega$ is a C^2-submanifold of dimension $n-1$ such that Ω lies locally on one side of $\partial\Omega$. By L we denote the strongly uniformly elliptic differential operator $Lu = -\sum_{i,j=1}^n a_{ij}D_iD_ju + \sum_{i=1}^n a_iD_iu + a_0u$, where $a_{ij} = a_{ji}$, $a_i, a_0 \in C(\bar{\Omega})$, and, adding αI, $\alpha > 0$, to both sides of (P_t), $a_0(x) \geqq \varepsilon > 0$ for all $x \in \bar{\Omega}$. B denotes either the Dirichlet boundary operator, or the operator $Bu = \partial u/\partial \nu + c_0 u$, where $\nu \in C^1(\partial\Omega, \mathbb{R}^n)$ is an outward pointing, nowhere tangent vectorfield on $\partial\Omega$ and $c_0 \in C^1(\partial\Omega, \mathbb{R})$, $c_0 \geqq 0$. Moreover

$$r \in C(\bar{\Omega})\backslash\{0\} \text{ satisfies } r(x) \geqq 0 \text{ in } \Omega. \tag{1}$$

Let λ_1 denote the principal eigenvalue of the linear BVP $Lu = \lambda u$ in Ω, $Bu = 0$ on $\partial\Omega$. Since $a_0 \geqq \varepsilon$ we have $\lambda_1 > 0$.

For the nonlinearity G we assume:

(G1) $G: \Delta \equiv \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous, continuously differentiable with respect to s and ξ and verifies the growth conditions

$$\text{(i)} \qquad |G(x, s, \xi)| \leqq c_1(|s|)(1 + |\xi|^2) \quad \text{for all} \quad (x, s, \xi) \in \Delta$$

$$\text{(ii)} \qquad |G(x, s, \xi)| \leqq c_2(s^+)(1 + |s| + |\xi|) \quad \text{for all} \quad (x, s, \xi) \in \Omega^0 \times \mathbb{R} \times \mathbb{R}^n$$

where $c_i: \mathbb{R}^+ \to \mathbb{R}^+$ are suitable increasing functions and Ω^0 denotes the set $\{x \in \Omega \mid r(x) = 0\}$ and $s^+ = s$ if $s \geqq 0$ and $s^+ = 0$ otherwise.

(G2) There exists a continuous function $G^*: \bar{\Omega}\times\mathbb{R}\to\mathbb{R}$ such that $G(x, s, \xi)\geqq G^*(x, s)$ for all $(x, s, \xi)\in\Delta$ and

(i) $$\limsup_{s\to-\infty} G^*(x, s)/s<\lambda_1$$

(ii) $$\liminf_{s\to+\infty} G^*(x, s)/s>\lambda_1$$

uniformly for $x\in\bar{\Omega}$.

Our main results are:

THEOREM 1. *Assume r and G satisfy* (1) *and* (G1), (G2), *respectively. Then there exists a* $t_0\in\mathbb{R}$ *such that* (P_t^*) *is solvable for* $t<t_0$ *and not solvable for* $t>t_0$.

THEOREM 2. *Assume the hypotheses of Theorem* 1 *are satisfied and in addition there exists for all bounded intervals* $\Gamma\subset\mathbb{R}$ *a constant* $M=M(\Gamma)$ *such that* $u_t(x)\leqq M$ *for all* $x\in\bar{\Omega}$, *where* u_t *is an arbitrary solution of* (P_t), $t\in\Gamma$. *Then there is a* $t_0\in\mathbb{R}$ *such that* (P_t) *possesses at least two solutions for* $t<t_0$, *at least one solution for* $t=t_0$ *and no solution for* $t>t_0$.

As an application we have:

COROLLARY. *If r and G satisfy* (1) *and* (G1), (G2), *respectively, and in addition* (G3) *for all* $m\in\mathbb{R}$ *there exists a constant* $M=M(m)$ *such that*

$$|G(x, s, \xi)|\leqq M(1+|s|+|\xi|)$$

for all $s\geqq m$, $x\in\bar{\Omega}$ *and* $\xi\in\mathbb{R}^n$, *then the hypotheses of Theorem* 2 *are satisfied.*

Some remarks concerning the comparison of our results with related former research are in order.

1. If $r(x)>0$ in Ω and $Bu=u$ we obtain a recent result of Kazdan and Kramer [**1**]. They prove a theorem of the type of Theorem 1 via the method of sub- and supersolutions. One should also mention the paper of Kazdan and Warner [**5**] in which similar problems for the case where G is not depending on the gradient have been studied. If $r(x)\geqq 0$ in Ω but $r\neq 0$, the basic step in [**1**] and [**5**], the construction of a supersolution for a suitable parameter value t does not carry over to our situation. Moreover no multiplicity result is obtained there and no assertion is made for $t=t_0$. Note however that for the multiplicity result we need an additional *a priori* estimate.

2. If $r\ngeqq 0$ we obtain an extension and sharpening of a result of Hess [**2**] where the nonlinearity was independent of the gradient. The technique developed in [**2**] does not apply to our situation if one has gradient-dependence.

3. In a recent paper Amann and Hess [**7**] proved a theorem of the type of Theorem 2 for the nonlinear BVP $Lu=f(x, u, t)$ in Ω, $Bu=0$ on $\partial\Omega$, where t denotes a real parameter value. If G is independent of the gradient and $r(x)>0$ in Ω their result is applicable to our situation. Hence, Theorem 2 and the corollary extend their result in some respect to the case where G is depending on the gradient and the function r satisfies $r\ngeqq 0$.

4. As a by-product we obtain a much simpler method for studying the equation $Lu=G(x, u, Du)$ in Ω, $Bu=0$ on $\partial\Omega$, if sub- and supersolutions $\bar{u}\leqq\hat{u}$ are given, as in the paper of Amann and Crandall [**4**]. Their method only allowed the

construction of an increasing fixed point operator on the order interval $[\bar{u}, \hat{u}]$. Our strongly increasing fixed point operator is a global one, which has some advantages, for example in studying multiplicity.

5. Questions similar to our problem have also been discussed by Dancer **[8]** for equations of the type $Lu = g(u) - f$.

Acknowledgment. I wish to thank M. Crandall and P. Hess for helpful suggestions.

II. A basic proposition

Fix $p > n$ throughout this paper. We will use the standard notation for the function spaces and denote by $C^1_B(\bar{\Omega})$ and $W^{2,p}_B(\Omega)$ the closed subspaces of $C^1(\bar{\Omega})$ and $W^{2,p}(\Omega)$ consisting of those functions which satisfy the boundary condition $Bu = 0$ on $\partial\Omega$. In the sequel we will need the following proposition.

PROPOSITION. *Assume $\gamma: C^1(\bar{\Omega}) \to L^\infty(\Omega)$ is a continuous operator which satisfies $|\gamma(u)(x)| \leqq c(|u(x)|)(1 + |Du(x)|^2)$ for all $x \in \bar{\Omega}$, where $c: \mathbb{R}^+ \to \mathbb{R}^+$ is a suitable increasing function and $\gamma(0) = 0$. Moreover assume γ admits for all $u, v \in C^1(\bar{\Omega})$ the representation*

$$\gamma(u) - \gamma(v) = \textstyle\sum_{i=1}^n b_i D_i(u - v) + b_0 \,.\, (u - v)$$

with $b_i = b_i(u, Du, v, Dv) \in L^\infty(\Omega)$, $i = 0, \ldots, n$, and $b_0 \geqq 0$. Then we have

(a) *The problem*

$$(*) \qquad Lu + \gamma(u) = f \quad \text{in } \Omega, \quad Bu = 0 \quad \text{on } \partial\Omega,$$

is for all $f \in L^\infty(\Omega)$ uniquely solvable in $W^{2,p}(\Omega)$ and the solution operator $H: L^\infty(\Omega) \to C^1_B(\bar{\Omega})$ is strongly increasing and compact where the function spaces are equipped with the natural order structures. If $S \subset L^\infty(\Omega)$ is a L^∞-bounded set, equipped with the induced L^p-topology given by the imbedding $L^\infty(\Omega) \to L^p(\Omega)$, then the operator $H_S: S \to C^1_B(\bar{\Omega})$: $f \to Hf$ is continuous.

(b) *If there exist an open subset $\Omega^* \subset \Omega$ and for all $v \in C^1(\bar{\Omega})$ a constant $M(v) \geqq 0$ such that $|\gamma(w + v)(x) - \gamma(v)(x)| \leqq M(v)(|w(x)| + |Dw(x)|)$ for $x \in \Omega^*$ and $w \leqq 0$, and if r satisfies* (1) *with $r(x) > 0$ for all $x \in \Omega \backslash \Omega^*$, then for all $a \in C(\bar{\Omega})$ and all $\phi \in C^1_B(\bar{\Omega})$ we find $T = T(a, \phi, r) \in \mathbb{R}$ such that $H(a + tr) \leqq \phi$ for all $t \leqq T$.*

Remark 1. Roughly speaking (b) says that linear growth of the operator $u \to \gamma(u)$ in u and $|Du|$ on the set $\{x \in \Omega \mid r(x) = 0\}$ implies that a solution of $(*)$ with $f = a + tr$ becomes negative if $t \in \mathbb{R}$ is small enough.

Remark 2. A typical example for such an operator is the following: Consider a continuous map $\beta: \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$: $(x, s, \xi) \to \beta(x, s, \xi)$ such that $\partial\beta/\partial s$ and $\partial\beta/\partial\xi$ exist and are continuous, which satisfies the growth conditions (G1) (i) and (ii), and in addition let $\partial\beta/\partial s \geqq 0$. Define $\gamma: C^1(\bar{\Omega}) \to L^\infty(\Omega)$ by $\gamma(u)(x) = \beta(x, u(x), Du(x))$. We find for $u, v \in C^1(\bar{\Omega})$

$$\gamma(u) - \gamma(v) = \sum_{i=1}^n \left(\int_0^1 (\partial\beta/\partial\xi_i)(\,.\,, v + t(u - v), Dv + tD(u - v))\, dt\right)(D_i u - D_i v)$$
$$+ \left(\int_0^1 (\partial\beta/\partial s)(\,.\,, v + t(u - v), Dv + tD(u - v))\, dt\right)(u - v)$$

which is an admissible representation.

For the proof of the proposition we need the following lemmata:

LEMMA 1. *Suppose* $b_i \in L^\infty(\Omega)$, $i=0,\ldots,n$, *and* $b_0(x) \geqq 0$. *Let* $u \in W^{2,p}(\Omega)$ *satisfy the inequalities*

$$Lu + \sum_{i=1}^{n} b_i D_i u + b_0 u \geqq 0 \quad \text{in } \Omega, \quad Bu \geqq 0 \quad \text{on } \partial\Omega.$$

Then $u \geqq 0$. *If* $u \neq 0$ *then* $u(x) > 0$ *for all* $x \in \Omega$ *and if* $u(x) = 0$ *for some* $x \in \partial\Omega$, *then* $(\partial u/\partial \nu)(x) < 0$ *where* ν *is an outward pointing vector at* x *which is not tangential.*

Lemma 1 follows from Bony's maximum principle [**6**].

LEMMA 2. *For every* $b \in L^\infty(\Omega)$ *there is exactly one solution* $u \in W^{2,p}(\Omega)$ *of the problem* $Lu = b(1+|Du|^2)$ *in* Ω, $Bu = 0$ *on* $\partial\Omega$. *Moreover there is an increasing function* $d: \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\|u\|_{2,p} \leqq d(\|b\|_\infty)$. *The function d is depending only on* L, Ω, p, *and* n.

The proof is a trivial variant of a result in [**4**].

Proof of the proposition. By the hypotheses on L and γ we find constants $M_1 < 0$ and $M_2 > 0$ such that the functions $\phi_i(x) = M_i$, $i = 1, 2$, are strict sub- and supersolutions of $(*)$. Let us denote by A the order interval $[\phi_1, \phi_2]$ in $C^1(\bar{\Omega})$. Assume $u, v \in W^{2,p}(\Omega)$ satisfy

$$Lu + \gamma(u) \geqq Lv + \gamma(v) \quad \text{in } \Omega, \quad Bu \geqq Bv \quad \text{on } \partial\Omega.$$

Using our representation of γ and Lemma 1 we immediately find that a solution of $(*)$ is unique and that the solution operator – if it exists – is strongly increasing. Moreover, the solution of $(*)$ has to be in the interior of A due to the fact that ϕ_1, ϕ_2 are strict sub- and supersolutions. Using Lemma 2 we obtain an *a priori* estimate for the solution u. In fact, the solution u of $(*)$ satisfies

$$Lu = b(1+|Du|^2) \quad \text{in } \Omega$$
$$Bu = 0 \quad \text{on } \partial\Omega,$$

where $b = (-\gamma(u)+f)(1+|Du|^2)^{-1}$. Since u has to be in $\operatorname{int}(A)$ we find by the growth condition of γ that $\|b\|_{L^\infty(\Omega)} \leqq$ const where the constant is only depending on ϕ_1 and ϕ_2. This implies by Lemma 2: $\|u\|_{C^1(\bar{\Omega})} \leqq$ const. Let B_R denote the open ball in $C^1(\bar{\Omega})$ with radius R and let $A_R = A \cap B_R$. By our preceding discussion one immediately obtains for R large enough that $\deg(I-P, A_R, 0) = 1$, where $P: A_R \to C^1(\bar{\Omega})$ is defined by $Pu = K(f-\gamma(u))$ and K denotes the solution operator of the linear BVP $Lu = g$ in Ω, $Bu = 0$ on $\partial\Omega$. ($K: L^\infty(\Omega) \to C^1(\bar{\Omega})$ compact). Up to now we have proved that $(*)$ is uniquely solvable for $f \in L^\infty(\Omega)$ and that the solution operator $H: L^\infty(\Omega) \to W_B^{2,p}(\Omega): f \to u$ is strongly increasing. Using Lemma 2 in the same way as before, we find that H maps L^∞-bounded sets into $W^{2,p}$-bounded sets. From the compactness of the imbedding $W_B^{2,p}(\Omega) \to C_B^1(\bar{\Omega})$ we deduce that $H: L^\infty(\Omega) \to C_B^1(\bar{\Omega})$ maps bounded sets into relatively compact sets and moreover H remains strongly increasing. Let $S \subset L^\infty(\Omega)$ be a L^∞-bounded set equipped with the L^p-topology. Assume $H_S: S \to C_B^1(\bar{\Omega})$ is not continuous. Then there exists a sequence $(f_n) \subset S$ converging in L^p to some $f \in S$ such that $\|Hf_n - Hf\|_{C^1(\bar{\Omega})} \geqq \delta$ for a suitable $\delta > 0$. Since (Hf_n) is bounded in

$W_B^{2,p}(\Omega)$ we may assume (for a subsequence) $Hf_n \to u$ strongly in $C_B^1(\bar{\Omega})$. Taking the limit in

$$LHf_n + \gamma(Hf_n) = f_n \quad \text{in } \Omega, \quad BHf_n = 0 \quad \text{on } \partial\Omega$$

we find

$$Lu + \gamma(u) = f \quad \text{in } \Omega, \quad Bu = 0 \quad \text{on } \partial\Omega.$$

Hence, by the uniqueness of solutions we must have $u = Hf$. This implies a contradiction: $0 < \delta \leqq \liminf \|Hf_n - Hf\|_{C^1(\bar{\Omega})} = 0$.

Now we prove (b). Let $a \in C(\bar{\Omega})$ and denote $H(a)$ by v. We have for $u = H(a + tr)$ and $w = u - v$

$$Lw + \gamma(w + v) - \gamma(v) = tr \quad \text{in } \Omega, \quad Bw = 0 \quad \text{on } \partial\Omega.$$

Let $\tilde{\gamma}\colon C^1(\bar{\Omega}) \to L^\infty(\Omega)\colon w \to \gamma(w+v) - \gamma(v)$ and denote by $\chi\colon \bar{\Omega} \to \mathbb{R}$ the characteristic function of $\Omega\backslash\Omega^*$. Define $\gamma^*\colon C^1(\bar{\Omega}) \to L^\infty(\Omega)$ by

$$\gamma^*(w)(x) = (\chi\tilde{\gamma}(w))(x) + M(v)w(x) - M(v)\,|Dw(x)|.$$

Obviously γ^* satisfies the hypotheses of part (a) and moreover we have for $w \leqq 0$

$$(\tilde{\gamma}(w) - \gamma^*(w))(x) = ((1-\chi)\tilde{\gamma}(w))(x) - M(v)w(x) + M(v)\,|Dw(x)|.$$

If $x \in \Omega^*$ we deduce

$$\begin{aligned}(\tilde{\gamma}(w) - \gamma^*(w))(x) &\geqq M(v)\,|w(x)| + M(v)\,|Dw(x)| - |\tilde{\gamma}(w)(x)| \\ &\geqq M(v)\,|w(x)| + M(v)\,|Dw(x)| - M(v)\,|w(x)| \\ &\qquad - M(v)\,|Dw(x)| \\ &\geqq 0.\end{aligned}$$

If $x \in \Omega\backslash\Omega^*$ we find

$$(\tilde{\gamma}(w) - \gamma^*(w))(x) = M(v)\,|w(x)| + M(v)\,|Dw(x)| \geqq 0.$$

Hence we have for all $w \in C^1(\bar{\Omega})$ with $w \leqq 0$: $\tilde{\gamma}(w) \geqq \gamma^*(w)$. Let us denote by H^* the solution operator of the following BVP

$$Lw + \gamma^*(w) = f \quad \text{in } \Omega, \quad Bw = 0 \quad \text{on } \partial\Omega.$$

Since $H^*(0) = 0$ and H^* is strongly increasing we must have that $w_0 \equiv H^*(-r) \ll 0$ in $C_B^1(\bar{\Omega})$ ($u \ll v$ means $v - u \in \operatorname{int}(P_{C_B^1(\bar{\Omega})})$, $P_{C_B^1(\bar{\Omega})} =$ positive cone in $C_B^1(\bar{\Omega})$). This implies the existence of a negative number $t' \in \mathbb{R}$ such that $-t'w_0 \ll \phi - v$. We find

$$\begin{aligned}L(-t'w_0) + \gamma^*(-t'w_0) &= -t'(Lw_0 + \gamma^*(w_0)) + (\gamma^*(-t'w_0) + t'\gamma^*(w_0)) \\ &= t'r + \chi(\tilde{\gamma}(-t'w_0) + t'\tilde{\gamma}(w_0)) \quad \text{in } \Omega \\ Bw_0 &= 0 \quad \text{on } \partial\Omega.\end{aligned}$$

Hence $\phi - v \gg -t'w_0 = H^*(t'r + b)$ with $b = \chi(\tilde{\gamma}(-t'w_0) + t'\tilde{\gamma}(w_0))$. We have for $t \leqq t'$

$$t'r + b \geqq tr + b \geqq 2tr - (tr - b)^+.$$

The set $((tr - b)^+)_{t \leqq t'}$ is L^∞-bounded by $\|b\|_\infty$ and converges in L^p to zero. By the continuity properties of H^* we find $\delta > 0$ such that for all $d \in L^\infty(\Omega)$, $\|d\|_\infty \leqq \|b\|_\infty$ and $\|d\|_{L^p(\Omega)} \leqq \delta$ we have

$$\phi - v \gg H^*(t'r + b + d) \quad \text{in } C^1_B(\bar{\Omega}).$$

For $t \leqq t'$ small enough we have $\|(tr-b)^+\|_{L^p(\Omega)} \leqq \delta$. This implies for $d = (tr-b)^+$

$$\begin{aligned}\phi - v &\gg H^*(t'r + b + (tr-b)^+)\\ &\geqq H^*(tr + b + (tr-b)^+)\\ &\geqq H^*(2tr - (tr-b)^+ + (tr-b)^+)\\ &= H^*(2tr).\end{aligned}$$

Let $T = 2t$. We obtain for $w^* = H^*(Tr) \quad (\leqq 0)$

$$\begin{aligned}Tr &= Lw^* + \gamma^*(w^*)\\ &\leqq Lw^* + \tilde{\gamma}(w^*)\\ &= L(w^*+v) + \gamma(w^*+v) - (Lv + \gamma(v))\\ &= L(w^*+v) + \gamma(w^*+v) - a \quad \text{in } \Omega.\end{aligned}$$

Hence

$$\begin{aligned}&L(w^*+v) + \gamma(w^*+v) \geqq a + Tr \equiv Lu + \gamma(u) \quad \text{in } \Omega\\ &B(w^*+v) = Bu = 0 \quad \text{on } \partial\Omega.\end{aligned}$$

From this we deduce

$$H(Tr + a) \leqq w^* + v \ll \phi - v + v = \phi$$

which completes the proof. ∎

Remark. 3. If the hypotheses of part (b) are not satisfied, one can not expect that the solutions of (*) are negative if t tends to $-\infty$. Consider the following example:

Let $\Omega = (0, 2)$. We study the following BVP

$$(**) \qquad -u'' + qu + \tilde{\gamma}(x, u) = f + tr \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

where $q = 4\exp(-2)$, $\tilde{\gamma}\colon \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\tilde{\gamma}(x, u) = -\exp(-2u + 4x) + \exp(4x) - q \min(u, 1 - \ln(2)),$$

and $f \in C(\bar{\Omega})$ by

$$f(x) = \begin{cases} \exp(4x) & x \in [0, 1]\\ 0 & x \in [3/2, 2]\\ \text{linear on} & (1, 3/2)\end{cases}$$

and $r \gneqq 0$ continuous with support in $(3/2, 2)$.

One easily verifies that the operator $\gamma\colon C^1(\bar{\Omega}) \to L^\infty(\Omega)\colon u \to \gamma(u)\colon \gamma(u)(x) = \tilde{\gamma}(x, u(x))$ is increasing, continuous and verifies the hypotheses of part (a). Let $v(x) = \ln(1-x) + 2x$ for $x \in [0, 1)$. We have $v(0) = 0$, $1 - \ln(2) = v(1/2) \geqq v(x)$ for all $x \in [0, 1)$ and $\lim_{x \to 1} v(x) = -\infty$. Denote the solution operator of (**) by H and let $u=H(f+tr)$. We obtain on $[0,1-z]$ for some z, $1/2>z>0$, small enough

$$-v''+qv+\tilde{\gamma}(x,v)=f=-u''+qu+\tilde{\gamma}(x,u) \quad \text{in } [0,1-z]$$
$$v(0)=u(0) \quad \text{and} \quad v(1-z)\leqq u(1-z).$$

By Lemma 1 we conclude that $v\leqq u$ on $[0,1-z]$. Hence

$$0<1-\ln(2)=v(1/2)\leqq u(1/2)=H(f+tr)(1/2)$$

for all $t\in\mathbb{R}$.

III. Proofs of the main results

Proof of Theorem 1. Define $\tilde{\gamma}:\bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ by $(G_s=\partial G/\partial s)$

$$\tilde{\gamma}(x,s,\xi)=\int_0^s(\operatorname{sign}(G_s(x,u,\xi)-|G_s(x,u,\xi)|)G_s(x,u,\xi))\,du+s.$$

Then the maps $s\to\tilde{\gamma}(x,s,\xi)$ and $s\to\tilde{\gamma}(x,s,\xi)+G(x,s,\xi)$ are strictly increasing for all fixed $(x,\xi)\in\bar{\Omega}\times\mathbb{R}^n$. Consider

$$\begin{aligned} Lu+\tilde{\gamma}(x,u,Du)&=G(x,v,Du)+\tilde{\gamma}(x,v,Du)+tr \quad \text{in } \Omega\\ Bu&=0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

where v is a given function in $C(\bar{\Omega})$. We will show that (2) is for all v uniquely solvable and that the solution operator $T:\mathbb{R}\times C(\bar{\Omega})\to C^1_B(\bar{\Omega})$: $(t,v)\to u(t,v)$ is strongly increasing in both arguments and compact. Define $\tilde{\gamma}_v:\bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ by

$$\begin{aligned}\tilde{\gamma}_v(x,s,\xi)=\tilde{\gamma}(x,s,\xi)&-G(x,v(x),\xi)-\tilde{\gamma}(x,v(x),\xi)\\ &+G(x,v(x),0)+\tilde{\gamma}(x,v(x),0).\end{aligned}$$

Then $\tilde{\gamma}_v$ is locally uniformly lipschitz continuous in s and ξ, satisfies $|\tilde{\gamma}_v(x,s,\xi)|\leqq c_v(|s|)(1+|\xi|^2)$ for a suitable increasing function $c_v:\mathbb{R}^+\to\mathbb{R}^+$ depending on v, and is increasing in s. Moreover we have

$$\begin{aligned}\tilde{\gamma}_v(x,s,\xi)-\tilde{\gamma}_v(x,s',\xi')&=\tilde{\gamma}_v(x,s,\xi)-\tilde{\gamma}_v(x,s',\xi)+\tilde{\gamma}_v(x,s',\xi)-\tilde{\gamma}_v(x,s',\xi')\\ &=b_0(s-s')+\sum_{i=1}^n b_i(\xi_i-\xi_i')\end{aligned}$$

where for $i=1\dots n$

$$b_i(x)=\begin{cases}(\tilde{\gamma}_v(x,s',\xi)-\tilde{\gamma}_v(x,s',\xi'))(\xi_i-\xi_i')^{-1} & \xi_i\neq\xi_i'\\ 0 & \text{otherwise}\end{cases}$$

and

$$b_0(x)=\begin{cases}(\tilde{\gamma}_v(x,s,\xi)-\tilde{\gamma}_v(x,s',\xi))(s-s')^{-1} & s\neq s'\\ 0 & \text{otherwise.}\end{cases}$$

Since $\tilde{\gamma}_v$ is strictly increasing in s we have $b_0(x)>0$ and by the lipschitz condition for $i=0\dots n$: $b_i\in L^\infty(\Omega)$. Let $\Omega^*=\Omega^0$. Since the mapping $(s,\xi)\to\tilde{\gamma}_v(x,s,\xi)$ is locally uniformly lipschitz continuous and satisfies for given $\tau\in\mathbb{R}^+$

$$|\tilde{\gamma}_v(x,s,\xi)|\leqq c_2(\tau)(1+|s|+|\xi|) \quad \text{for all} \quad (x,s,\xi)\in\Omega^*\times\mathbb{R}\times\mathbb{R}^n \quad \text{with} \quad s\leqq\tau,$$

we find for all given $w \in C^1(\bar{\Omega})$ a constant $M(w)$ such that

$$|\tilde{\gamma}_v(x, s+w(x), \xi+Dw(x)) - \tilde{\gamma}_v(x, w(x), Dw(x))| \leqq M(w)(|s|+|\xi|)$$
$$\text{for all} \quad (x, s, \xi) \in \Omega^* \times \mathbb{R}^- \times \mathbb{R}^n. \quad (3)$$

Let us define the operator $\gamma_v: C^1(\bar{\Omega}) \to L^\infty(\Omega)$ by $\gamma_v(u) = \tilde{\gamma}_v(., u, Du)$. By (3) we can find for all $w \in C^1(\bar{\Omega})$ a constant $M(w) > 0$ such that

$$|(\gamma_v(u+w) - \gamma_v(w))(x)| \leqq M(w)(|u(x)| + |Du(x)|)$$
$$\text{for all} \quad x \in \Omega^* \quad \text{and all} \quad u \in C^1(\bar{\Omega}) \quad \text{with} \quad u \leqq 0.$$

The preceding discussion of $\tilde{\gamma}_v$ implies that γ_v satisfies the hypotheses of the proposition.

This implies that (2) is uniquely solvable because it is equivalent to

$$Lu + \gamma_v(u) = G(., v, 0) + \tilde{\gamma}_v(., v, 0) + tr \quad \text{in } \Omega \quad Bu = 0 \quad \text{on } \partial\Omega. \quad (4)$$

Let $T: \mathbb{R} \times C(\bar{\Omega}) \to C^1_B(\bar{\Omega}): (t, v) \to u(t, v)$ be the solution operator of (4). Using Lemma 2 and proceeding as in the proof of the proposition, part (a), we deduce that T is strongly increasing and compact. The solution u of

$$Lu + \gamma_0(u) = G(., 0, 0) + \tilde{\gamma}_0(., 0, 0) + tr \quad \text{in } \Omega \quad Bu = 0 \quad \text{on } \partial\Omega$$

is by definition $T(t, 0)$. Applying now the proposition, we conclude for $t^* \leqq 0$ small enough $T(t^*, 0) \ll 0$. From results in [**1**] (Proposition 2.13) we obtain because of the asymptotic behaviour of G as $s \to -\infty$, that there exists a strict subsolution $\bar{u} \leqq 0$ of (P_{t^*}) which implies $T(t^*, \bar{u}) \gg \bar{u}$ in $C^1_B(\bar{\Omega})$. Since $T(t^*, .)$ is strongly increasing, it maps $V \equiv [\bar{u}, 0]_{C_B{}^1(\bar{\Omega})} \to \operatorname{int}(V)$. Moreover V is bounded in $C(\bar{\Omega})$ which implies that $T(t^*, V)$ is bounded in $C^1_B(\bar{\Omega})$. Hence, by the compactness of $T(t^*, .): C(\bar{\Omega}) \to C(\bar{\Omega})$, we find a fixed point in $\operatorname{int}(V)$ by Schauder's fixed point theorem. Let $t_0 \equiv \sup\{t \in \mathbb{R} \mid (P_t) \text{ is solvable}\}$. If (P_t) has a solution u_t for some t we infer that u_t is a strict supersolution for all $t' < t$. As above we find a strict subsolution $\bar{u} \leqq u_t$, which implies the existence of a solution $u_{t'} \in \operatorname{int}([\bar{u}, u_t]_{C_B{}^1(\bar{\Omega})})$ of $(P_{t'})$. By the asymptotic behaviour of G as $s \to +\infty$, one can show that $t_0 < +\infty$ (see [**1**, proof of Theorem 3.4, p. 636]).

Proof of Theorem 2. Let T be as in Theorem 1 and choose $t < t_0$. As we have seen in the proof above there exist strict sub- and supersolutions of (P_t): $\bar{u} \leqq \hat{u}$. Denote the order interval $[\bar{u}, \hat{u}]_{C_B{}^1(\bar{\Omega})}$ by A. Since $T = T(t, .)$ maps $A \to \operatorname{int}(A)$ we find that the fixed points of T in A are in $\operatorname{int}(A)$. We may assume that there is only one fixed point $u \in A$ (otherwise we are done). Finally we find for some $\varepsilon > 0$ such that $u + \varepsilon\mathbb{B} \subset \operatorname{int}(A)$, where $\mathbb{B}$ denotes the open unit ball in $C^1_B(\bar{\Omega})$, making use of the standard properties of the Leray–Schauder-Degree and the fixed point index

$$\deg(I - T, u + \varepsilon\mathbb{B}, 0) = i(T, u + \varepsilon\mathbb{B}, C^1_B(\bar{\Omega})) = i(T, u + \varepsilon\mathbb{B}, A)$$
$$= i(T, A, A) = 1.$$

For a solution u of $T(\bar{t}, u) = u$, $\bar{t} \in [t, t_0 + 1] \equiv \Gamma$, we have $u(x) \leqq M(\Gamma)$. On the other hand solutions are bounded from below by

$$\begin{aligned} Lu &= G(x, u, Du) + tr \\ &\geqq G^*(x, u) + tr \\ &\geqq (\lambda_1 - \bar{\varepsilon})u + h \quad \text{in } \Omega \\ Bu &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

for some $\bar{\varepsilon} > 0$ small enough and a suitable constant h depending only on $\bar{\varepsilon}$ and Γ. Hence $u \geqq K(h)$, where K denotes the positive solution operator of

$$Lu - (\lambda_1 - \bar{\varepsilon})u = g \quad \text{in } \Omega, \quad Bu = 0 \quad \text{on } \partial\Omega.$$

Since the solutions of $(P_{\bar{t}})$, $\bar{t} \in \Gamma$, are contained in a $C(\bar{\Omega})$-bounded set, they must be bounded in $C_B^1(\bar{\Omega})$ by a constant $k-1$. We deduce, since we may assume $u + \varepsilon\mathbb{B} \subset k\mathbb{B}$, making use of the additivity, homotopy and excision properties of the Leray–Schauder-Degree

$$\begin{aligned} \deg(I - T, k\mathbb{B} \backslash u + \varepsilon\mathbb{B}, 0) &= \deg(I - T, k\mathbb{B}, 0) - 1 \\ &= \deg(I - T(t_0 + 1, .), k\mathbb{B}, 0) - 1 \\ &= -1 \end{aligned}$$

which implies the existence of a second solution. To study (P_{t_0}) choose a sequence (t_n), $t_n < t_0$, $t_n \to t_0$. One can show as before that the sequence (u_n) of solutions u_n of (P_{t_n}) is relatively compact in $C_B^1(\bar{\Omega})$. Hence we may assume (for a subsequence) $u_n \to u$ strongly in $C_B^1(\bar{\Omega})$. Taking the limit for $T(t_n, u_n) = u_n$ we find $T(t_0, u) = u$ which completes the proof. ■

Proof of the Corollary. It is enough to show that for all bounded intervals $\Gamma \subset \mathbb{R}$ there exists a constant $M = M(\Gamma)$ such that a solution u of (P_t), $t \in \Gamma$, satisfies $u(x) \leqq M$ for all $x \in \bar{\Omega}$. As in the proof of Theorem 2 one can show the existence of a constant M^*: u is solution of (P_t), $t \in \Gamma$, then $u(x) \geqq -M^*$ for all $x \in \bar{\Omega}$. Assume there exist sequences (t_n) and $(u_n) \subset W^{2,p}(\Omega)$ with $\|u_n\|_\infty \to \infty$ verifying

$$Lu_n = G(x, u_n, Du_n) + t_n r \quad \text{in } \Omega, \quad Bu_n = 0 \quad \text{on } \partial\Omega. \tag{5}$$

Since $\|u_n\|_\infty \to \infty$ we have $\|u_n\| = \|u_n\|_{C_B{}^1(\bar{\Omega})} \to \infty$. For $w_n = u_n/\|u_n\|$ we obtain

$$\begin{aligned} Lw_n &= (G(x, u_n, Du_n) + t_n r)/\|u_n\| \\ &\geqq (G^*(x, u_n) + t_n r)/\|u_n\| \\ &\geqq (\lambda_1 + \varepsilon)w_n - \eta/\|u_n\| \end{aligned} \tag{6}$$

for suitable constants $\varepsilon > 0$ and $\eta > 0$ depending on Γ. By the boundedness of the set $((G(x, u_n, Du_n) + t_n r)/\|u_n\|)$ (Condition (G3)) in $C(\bar{\Omega})$ we infer the relative compactness of (w_n) in $C_B^1(\bar{\Omega})$. Hence we may assume $w_n \to w$ in $C_B^1(\bar{\Omega})$ for some $w \in C_B^1(\bar{\Omega})$ having $\|w\| = 1$. Since (u_n) is bounded from below we have $w \gneqq 0$. Thus we deduce from (11) taking the limit

$$(\lambda_1 + \varepsilon)^{-1} w - \bar{K}w = y \geqq 0 \tag{7}$$

where $\bar{K}$ denotes the solution operator of the linear BVP $Lu = g$ in Ω, $Bu = 0$ on $\partial\Omega$, where g is a given function in $L^p(\Omega)$. Since $\bar{K}$: $C_B^1(\bar{\Omega}) \to C_B^1(\bar{\Omega})$ is a strongly positive compact endomorphism and the spectral radius is $r(\bar{\bar{K}})=\lambda_1^{-1}$ we conclude, observing that $(\lambda_1+\varepsilon)^{-1}<r(\bar{K})$, that equation (7) cannot have a positive solution (see [**3**, Theorem 3.2]). This contradiction proves the corollary. ■

References

1 J. L. Kazdan and R. J. Kramer. Invariant criteria for existence of solutions to second order quasilinear elliptic equations. *Comm. Pure Appl. Math.* **31** (1978), 619–645.

2 P. Hess. On a nonlinear elliptic boundary value problem of the Ambrosetti-Prodi type. *Boll. Un. Mat. Ital.* **17** (1980), 187–192.

3 H. Amann. Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. *SIAM Rev.* **18** (1976), 620–709.

4 H. Amann and M. Crandall. On some existence theorem for semilinear elliptic equations. *Indiana Univ. Math. J.* **27** (1978), 779–790.

5 J. L. Kazdan and F. W. Warner. Remarks on some quasilinear elliptic equations. *Comm. Pure Appl. Math.* **28** (1975), 567–597.

6 J. M. Bony. Principe du maximum dans les espace de Sobolev. *C.R. Acad. Sci. Paris Ser. A* **265** (1967), 333–336.

7 H. Amann and P. Hess. A multiplicity result for a class of elliptic boundary value problems. *Proc. Roy. Soc. Edinburgh Sect. A* **84** (1979), 145–151.

8 E. N. Dancer. On the ranges of certain weakly nonlinear elliptic partial differential equations. *J. Math. Pures Appl.* **57** (1978), 351–366.

(*Issued* 11 *March* 1981)